COMBINATORICS OF FINITE TYPE CLUSTER SEEDS WITH EMPHASIS ON TYPE A_n

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Abstract. We provide a combinatorial overview of finite geometric type cluster algebras, with an emphasis on type A_n. We begin with a summary of the basic theory leading to a construction of a cluster algebra. We repeat the finite type classification as in [2], and discuss the classification of finite mutation type quivers. We discuss the associahedra, a family of convex polytopes which model the combinatorics of type A_n seeds, and conclude with a discussion of triangulations and a proof of a combinatorial formula for the associahedra.

Contents

1. Introduction

Cluster algebras, introduced by Sergey Fomin and Andrei Zelevinsky in their influential 2001 paper [1] are a relatively new class of commutative rings, equipped with a distinguished set X of generators, called *cluster variables*. This set is obtained by joining together n-tuples of cluster variables, called *clusters*. The relationship between clusters is as follows: To each cluster $\mathbf{x}$, we assign a matrix B, called an *exchange matrix*. The pair $(\mathbf{x}, B)$, called a *seed*, is then transformed like so: A cluster variable $x = \mathbf{x}_i$ is chosen, and one obtains another cluster $\mathbf{x}'$ by replacing x with a variable x', related via the following *exchange relation*:

$$xx' = M_1 + M_2, \tag{1.1}$$

where M_1 and M_2 are monomials without common divisors, determined by the exchange matrix B and the $n-1$ remaining variables. We also transform the exchange matrix B through a process called *mutation*, replacing it with a matrix B' as defined

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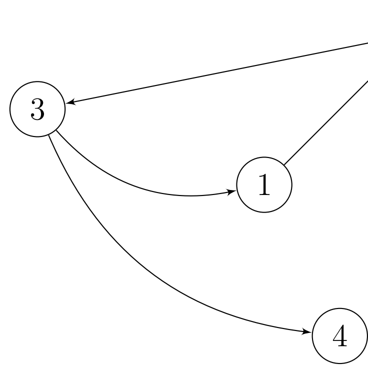

FIGURE 1. A quiver.

in Section 2. Imposing that any two clusters may be obtained via seed exchanges fixes a combinatorial structure on the generators, which are of interest in this paper.

Typically, the set of clusters obtained by exchanges is infinite. A subclass of the cluster algebras are *cluster algebras of finite mutation type*, in which this set is finite. A natural construction from this set is the *exchange graph*, whose vertices are the clusters, and where two vertices share an edge if and only if one may be obtained from the other via a single exchange. The exchange graph may be viewed as the one-skeleton of a simplicial complex, whose ground set is X (the set of all cluster variables) and whose maximal simplices are the clusters. For a large portion of the finite type cluster algebras, this simplicial complex may be realised as a convex polytope.

This paper is a combinatorial overview of this theory. We begin with a review of most of the fundamental notions of cluster algebra theory. We then focus our attention on the cluster algebras of type A_n, who are naturally associated with triangulations of convex polygons, and hence may be associated to an abstract polytope called an associahedron. We then derive a matrix that enumerates how these polytopes inherit each other, and discuss some of its consequences.

2. Quiver mutation and cluster algebras

In this section, we begin to walk through the construction of a cluster algebra. A *quiver* is a directed graph on finitely many vertices. The combinatorial structure of exchange relations may be expressed as *cluster quivers*, which are quivers without loops or 2-cycles. From now on, quivers are always cluster quivers.

Definition 2.1. A *quiver* is a finite directed graph without loops or 2-cycles.

To every quiver Q on n vertices, there exists an $n \times n$ matrix B, called the *incidence matrix* of Q. This matrix encodes the adjacencies of the vertices of Q, whose (i, j)th entry is the number of (signed) incidences of arrows with source i and target j. It is easy to see that this matrix is skew-symmetric, that is $b_{ij} = -b_{ji}$. We use the notation B again for this matrix, as it is precisely the incidence matrix of a quiver that we use for the exchange relations of cluster variables.

We now turn to the notion of *quiver mutation*.

Definition 2.2. Quiver mutation. Let $\mathcal{Q}_n$ be the set of all quivers on n vertices. Then define $\mu_k : \mathcal{Q}_n \to \mathcal{Q}_n$, called *mutation on the kth vertex*, by transforming $Q \in \mathcal{Q}_n$ as follows:

(1) For every path of arrows $i \to k \to j$, add a new arrow $i \to j$.
(2) Reverse any arrows incident or outbound from k.
(3) Delete a maximal collection of 2-cycles that appear after executing step 1.

This definition can be stated equivalently in terms of how the incidence matrix transforms.

Definition 2.3. Matrix mutation. Let B be the $n \times n$ incidence matrix associated to $Q \in \mathcal{Q}_n$. Let $[b_{ik}b_{kj}]_+$ denote the positive part of $b_{ik}b_{kj}$. Then *mutation at* k on B, denoted $\mu_k(B)$, is the $n \times n$ matrix B' defined as follows:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{for } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \, [b_{ik}b_{kj}]_+ \,, & \text{otherwise.} \end{cases} \tag{2.1}$$

Since quivers and their incidence matrices are equivalent data, we may jointly refer to quiver and matrix mutation as simply mutation. We now list some properties of mutation:

Proposition 2.4. *Mutation satisfies the following:*

(1) *Mutation is involutive, that is $\mu_k^2(Q) = Q$.*
(2) *Mutation preserves the rank of the incidence matrix.*

By repeatedly mutating at vertices, we may generalise Definition 2.2.

Definition 2.5. Mutation at a sequence of vertices. Let $k_1, k_2, ..., k_m$ be a sequence of m vertices. We call the composition $\mu_{k_m} \circ \mu_{k_{m-1}} \circ ... \circ \mu_{k_1}$ *mutation on a sequence of vertices*, and denote this function by $\mu_{k_1,k_2,...,k_m}$.

Typically it happens that a sequence of mutations produces quivers which are identically structured, but may differ only up to labelling. Any two such quivers have the same combinatorial structure, and thus we typically consider quivers up to isomorphism.

Definition 2.6. Quiver isomorphism. Two quivers Q_1, Q_2 are called *isomorphic* if there exists a directed graph isomorphism between them. We then write $Q_1 \cong Q_2$.

Generally we are interested in what quivers may be obtained just by applying mutations to a given starting quiver.

Definition 2.7. Mutation equivalence. Let Q_1, Q_2 be quivers. We say that Q_1 is *mutation equivalent* to Q_2 if there exists a sequence of vertices $k_1, k_2, ..., k_m$ such that $\mu_{k_1,k_2,...,k_m}(Q_1) \cong Q_2$.

Mutation equivalence may be verified to be an equivalence relation on $\mathcal{Q}_n$; it is reflexive and symmetric by item 1. of Proposition 2.4, and transitivity is a simple consequence too; if $\mu_{k_1,k_2,...,k_n}(Q_1) \cong Q_2$ and $\mu_{l_1,l_2,...,l_m}(Q_2) \cong Q_3$, then clearly

$\mu_{k_1,k_2,\ldots k_n,l_1,l_2,\ldots l_m}(Q_1) \cong Q_3$. Given a fixed quiver $Q \in \mathcal{Q}_n$, we may consider all quivers mutation-equivalent to Q.

Definition 2.8. Mutation class. The *mutation class* $[Q]$ of Q is the equivalence class of Q under mutation-equivalence in $\mathcal{Q}_n$.

In the frame of a quiver's mutation class, we may construct the exchange graph discussed in the introduction.

Definition 2.9. Exchange graph. The *exchange graph* $\mathcal{G}(Q)$ of a quiver $Q \in \mathcal{Q}_n$ is the graph whose vertex set is $[Q]$, and where any two vertices share an edge if they are isomorphic up to a single mutation.

We obtain a larger class of exchange matrices if we weaken our skew-symmetrical constraint.

Definition 2.10. Skew symmetrisable. We say that a matrix B is *skew-symmetrisable* if there exists a diagonal matrix D with positive integer diagonal entries such that the matrix DB is skew-symmetric. We then refer to D as the *symmetriser* of B.

One may verify that a skew-symmetrisable matrix remains skew-symmetrisable under mutations.

We need one final extension to our set up before we present the definition of a cluster algebra.

Definition 2.11. Seed. Let B be an $n \times n$ skew-symmetrisable matrix, let k be a field of characteristic 0 and $\mathcal{F}$ a field extension, and let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be a tuple of n algebraically independent elements in the ambient field $\mathcal{F}$. A *seed* is a pair $(\mathbf{x}, B)$.

We may choose a fixed seed S and construct a cluster algebra from it as follows.

Definition 2.12. Seed mutation. Let $S = (\mathbf{x}, B)$ be a seed. We define *mutation on S at k*, denoted μ_k, as the operation that produces the seed $S' = (\mathbf{x}', B')$ defined as follows:

(1) $B' = \mu_k(B)$ as usual matrix mutation.
(2) $\mathbf{x}' = (x_1, ..., x'_k, ..., x_n)$, where x'_k satisfies

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{jk}<0} x_j^{-b_{jk}}.$$

We want to consider seeds up to isomorphism when constructing a cluster algebra; we say two seeds are *isomorphic* if they can be obtained from each other by a simultaneous permutation of cluster variables and matrix rows/columns.

Definition 2.13. Cluster algebra. The *cluster algebra* $\mathcal{A}(\mathbf{x}, B)$ is the subalgebra of the ambient field $\mathcal{F}$ generated by the set

$$X(\mathbf{x}, B) = \bigcup_{(\mathbf{x}',B')\sim(\mathbf{x},B)} \{x'_1, x'_2, ..., x'_n\}.$$

An extension of the combinatorial set up will be useful in later sections.

Definition 2.14. Extended exchange matrices. Fix $m \geq n \geq 1$. Let $\widetilde{B}$ be an $m \times n$ matrix with integer entries. Write $\widetilde{B}$ in block form as

$$\widetilde{B} = \begin{bmatrix} B \\ C \end{bmatrix},$$

where B is an $n \times n$ matrix and C is an $(m-n) \times n$ matrix. We call B the *principal part* of $\widetilde{B}$. We say that $\widetilde{B}$ is an *exchange matrix* if the principal part is skew-symmetrisable.

We label the indices $k \in \{1, ..., n\}\}$ *mutable* and the indices $k \in \{n+1, ..., n+m\}$ *frozen*.

3. Finite mutation type seeds

In this section, finite type cluster algebras and quivers are completely classified, and their combinatorial structure discussed. The finite type cluster algebra classification is in a one-to-one correspondence to the classification of the finite crystallographic root systems, an astonishing result due to Fomin-Zelevinsky [2]. Moreover, the property of the initial seed of a cluster algebra $(\mathbf{x}, \mathbf{y}, B)$ to be of finite type does not depend on the choice of coefficients $\mathbf{y}$, and only on the exchange matrix B. We say that a cluster algebra is of *finite type* if its initial seed mutation class is finite, or equivalently that the cluster algebra has finitely many cluster variables. We call this seed mutation class a *seed pattern.* Recall that a quiver is of *finite mutation type* if its mutation class is finite. Not every finite mutation type seed is of finite type; the Markov quiver has singleton mutation class, but infinite seed mutation class.

We first state some equivalent conditions for a seed to be of finite type. We borrow some notation from [3].

Lemma 3.1. *A cluster algebra is of finite type if and only if all of its exchange matrices $B = (b_{ij})$ have the property that $|b_{ij}b_{ji}| \leq 3$ for any pair of indices i, j.*

Proof. Suppose on the contrary that there exists a matrix $B'_{ij} = (b'_{ij})$ with indices k, l such that $|b'_{kl}b'_{lk}| > 3$. Then by iterative mutation on k and l, we can make the product $|b'_{kl}b'_{lk}|$ arbitrarily large, and thus generate arbitrarily many distinct seeds. □

The above lemma motivates the following definition.

Definition 3.2. 2-finite. A skew-symmetrisable matrix $B = (b_{ij})$ is called 2-*finite* if and only if for any matrix B' mutation equivalent to B and for any indices i and j, we have $|b'_{ij}b'_{ji}| \leq 3$.

Lemma 3.3. *A seed pattern is of finite type if and only if it has finitely many cluster variables.*

Proof. Obviously if a seed pattern has finitely many seeds, it must have finitely many cluster variables. Suppose now that a seed pattern haws finitely many cluster variables. Suppose for contradiction that there are infinitely many seeds; then the possible entries of the exchange matrices of these seeds must be unbounded. But then one of these exchange matrices must be 2-finite, and hence on the associated seed there exists a sequence of mutations that produce infinitely many cluster variables, contradicting our assumption. □

We now introduce the Cartan matrices of finite type, whose classification is what our classification of finite type cluster algebras is parallel to.

Definition 3.4. Cartan matrices. A square $n \times n$ integer matrix $A = (a_{ij})$ is called a *symmetrisable generalised Cartan matrix* if it satisfies the following conditions:

(1) $a_{ii} = 2$ for all $1 \leq i \leq n$.
(2) $a_{ij} \leq 0$ for all $i \neq j$.
(3) There exists a diagonal matrix D with positive diagonal entries such that DA is symmetric.

We call D the *symmetriser* of A. We call A *positive* if DA is positive definite. Any such matrix satisfies $a_{ij}a_{ji} \leq 3$ for $i \neq j$. We refer to positive symmetrisable generalised Cartan matrices as *Cartan matrices of finite type.*

The Cartan matrices encode information about the geometry of root systems. The classification of Cartan matrices of finite type is in a direct correspondence to the classification of finite crystallographic root systems. To each Cartan matrix $A = (a_{ij})$, we can assign a graph, called the *Dynkin diagram* of A, which is a graph with vertices $1, ..., n$ and where vertices i, j with $i \neq j$ are joined by $a_{ij}a_{ji}$ edges, with arrows according to how the corresponding root vectors are arranged [5]. Each exchange matrix can be associated to a Cartan matrix of finite type.

Definition 3.5. Cartan counterpart. Let $B = (b_{ij})$ be a skew-symmetrisable integer matrix. Define $A = A(B) = (a_{ij})$ by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

We call A the *Cartan counterpart* to B.

We are now ready to present the classification of cluster algebras of finite type, due to S. Fomin and A. Zelevinsky in [2]:

Theorem 3.6. *A cluster algebra is of finite type if and only if it has a seed with exchange matrix B, such that the Cartan counterpart $A(B)$ is a Cartan matrix of finite type.*

Similarly to how the Cartan-Killing classification partitions semi simple Lie algebras into 'types' based on the type of their Cartan matrices, so the finite type cluster algebras may be given types.

We may follow a similar course to classify which quivers are mutation finite. Felikson, Shapiro and Tumarkin provide an answer in [6]:

Theorem 3.7. *A quiver Q is of finite mutation type if and only if it is one of the following:*

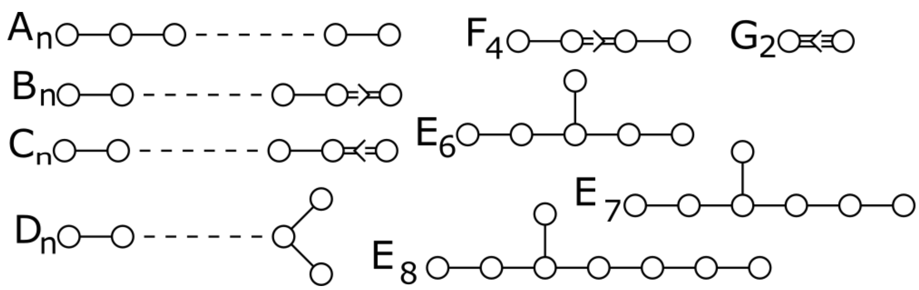

Figure 2. Dynkin diagrams corresponding to all the possible Cartan counterparts of finite type cluster algebras.

(1) *A quiver with two vertices.*
(2) *A quiver associated to a triangulation of a two-dimensional bordered surface* [7].
(3) *A quiver that is mutation equivalent to one of a list of 11 exceptional quivers.*

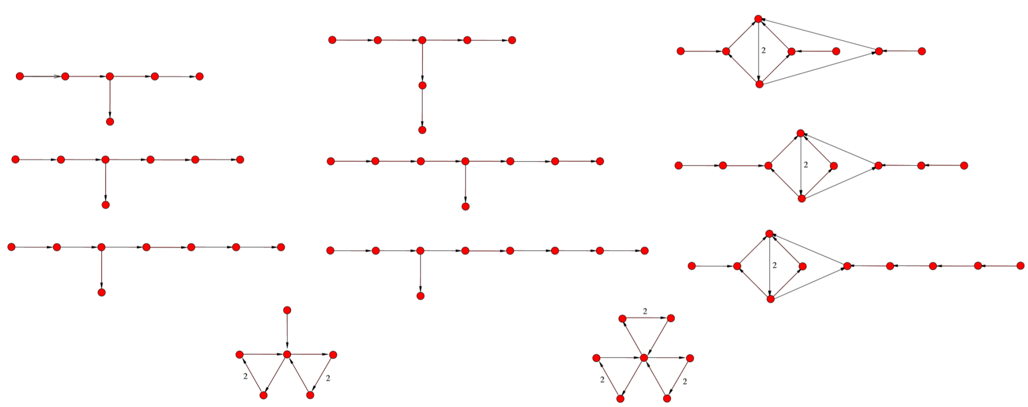

Figure 3. The 11 exceptional type quivers.

Of particular interest are the type A_n quivers, because these may be assigned to triangulations of regular polygons (Section 5). Also of interest is the type X_6 quiver. The exchange graph in Figure 3 admits plenty of symmetries, and the mutation process interacts nicely with the automorphism group of each of the quivers in the mutation class.

4. Associahedra

The combinatorics of type A_n seeds are governed by the associahedra (or Stasheff polytopes, named after James Stasheff [8].) We present a summary of these polytopes.

There are many ways to construct the associahedra, the most well-known of these ways as the secondary polytope of a regular polygon [4]. We summarise the combinatorial data attached to the associahedra; so we specialise to the abstract polytope

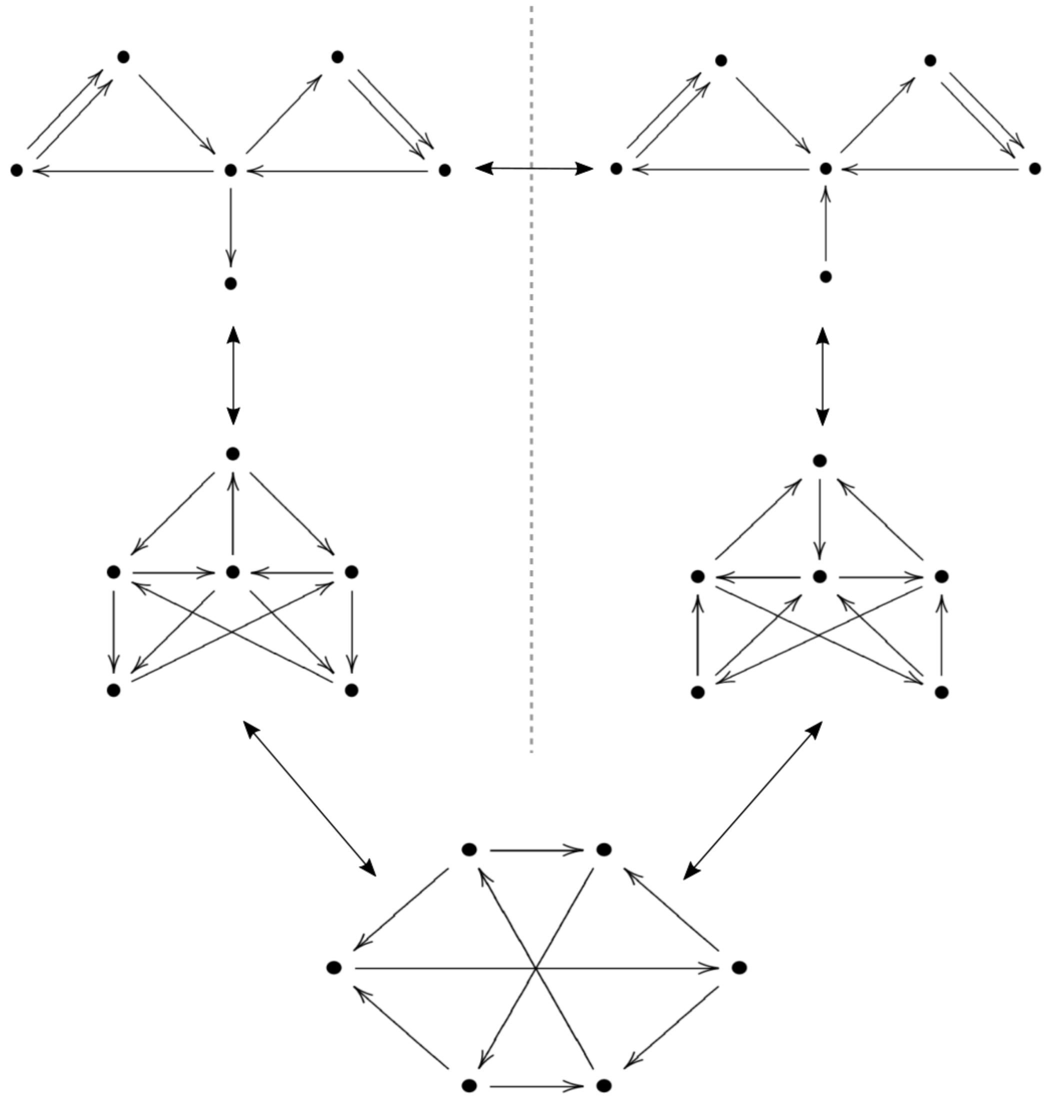

FIGURE 4. The exchange graph of X_6. Notice the symmetry about the dashed line obtained by reversing arrows.

associated to the flip graph of a triangulation of a regular polygon. See Section 5 for the details.

Definition 4.1. Associahedron Let $n \geq 1$ be an integer. Let P_{n+1} be a regular polygon with $n+1$ vertices. We define K_n, the $n-$*associahedron*, as the abstract polytope associated to the flip graph of a triangulation of P_{n+1}.

We first present the main features of these polytopes. Proofs are available in [4]. By exploiting known numerics about triangulations of regular polygons, we can extract information about the associahedra.

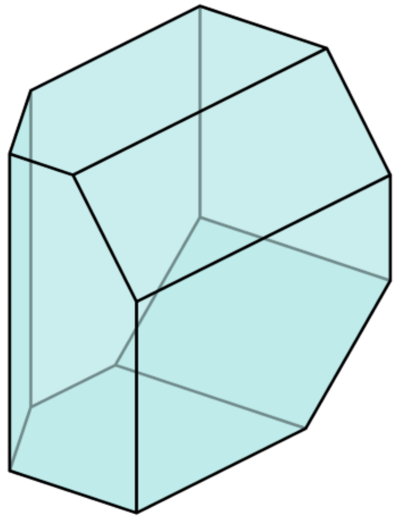

FIGURE 5. The associahedron K_5.

Lemma 4.2. *K_n consists of C_{n-1} vertices, where C_n is the nth Catalan number. It is $(n-2)-$regular, that is each vertex of K_n connects to exactly $n-2$ neighbouring vertices by edges. It consists of $(n-2)C_{n-1}/2$ edges.*

The associahedra have a recursive structure; lower-dimensional associahedra manifest in higher-dimensional ones. We first need to define precisely what we mean by this.

Definition 4.3. Inheritance. Fix integers $n \geq k$. We say that K_k is *inherited* in K_n if we may obtain a copy of K_k by restricting the vertex set of K_n. We count two inheritances as indistinct if they use the same vertices of K_n.

There may be different ways for an inheritance to occur; see for example Figure 6. The following records how many times this happens.

Theorem 4.4. *Fix integers $n \geq k$. The number of times the associahedron K_k is distinctly inherited in K_n is given by the inheritance matrix*

$$s_k^n = (n+1) \sum_{L \in P_{n-k}} (k)_{|L|-1} \Delta_L, \tag{4.1}$$

where P_{n-k} is the set of partitions of $n-k$, $(x)_n = \frac{x!}{(x-n+1)!}$ denotes the falling factorial of x up to n, and

$$\Delta_L = \prod_{l \in L} \frac{C_l}{\mathbf{R}(l)!} \tag{4.2}$$

where C_l is the lth Catalan number and $\mathbf{R}(l)$ is the number of parts of length l in L.

Proof. Deferred to Section 5, where we consider triangulations. $\square$

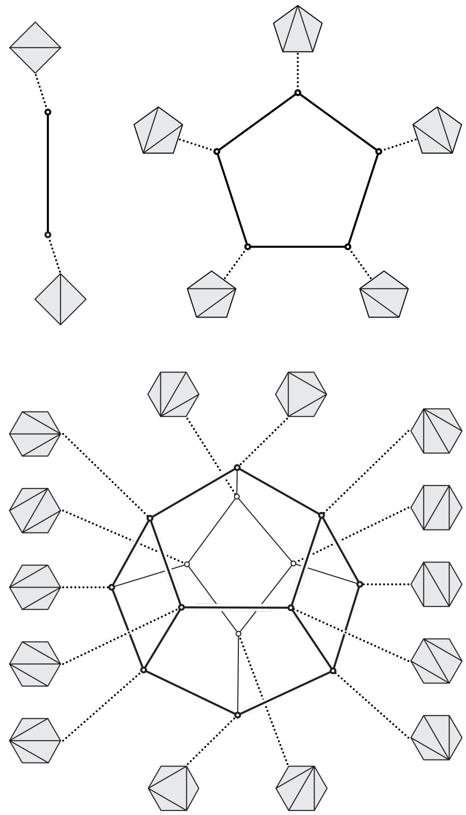

FIGURE 6. Some low-dimension Stasheff polytopes, and the corresponding flip graphs that generate them.

An interesting result concerning these polytopes concerns the Lagrange inversion theorem, due to Loday [9].

Theorem 4.5. *Let $f(x) = x + a_1x^2 + a_2x^3 + ... + a_nx^{n+1} + ...$ be a formal power series in the variable x, and let $g(x) = x + b_1x^2 + b_2x^3 + ... + b_nx^{n+1} + ...$ be its inverse under left composition, that is we impose $g(f(x)) = x$. This allows us to write the coefficients b_n as polynomials in $a_1, ..., a_n$. We have the following result:*

$$b_n = \sum (-1)^{\sum n_i} \lambda(n_1, ..., n_k) a_1^{n_1} ... a_n^{n_k}, \tag{4.3}$$

where the sum is extended to all k-tuples of integers $(n_1, ..., n_k)$ so that $n_1 + 2n_2 + ... + kn_k = n$, and the coefficient $\lambda(n_1, ..., n_k)$ is the number of cells of the Stasheff polytope K_{n-1} that are isomorphic to the cartesian product $(K_0)^{n_1} \times ... \times (K_{k-1})^{n_k}$.

5. Triangulations and type A_n seeds

In this section, we discuss the useful relationship between type A_n seeds and triangulations of regular $(n+3)-$gons. In the process, we obtain a proof for formula (4.1).

Definition 5.1. Triangulation. Let P_n be a regular $n-$gon. Then a *triangulation* $\mathcal{T}$ of P_n is a maximal collection of non-intersecting diagonals of P_n.

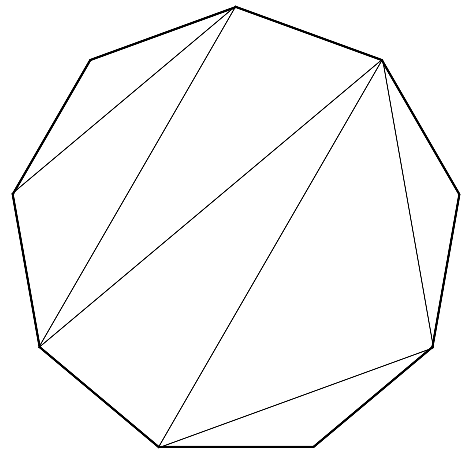

FIGURE 7. A triangulation of a k-gon.

We call these diagonals *arcs* of the triangulation. When a triangulation $\mathcal{T}$ has been fixed, we can assign a quiver to it as follows: The vertices are the arcs of $\mathcal{T}$, and we draw an arrow from arc i to arc j if and only if i and j are edges of the same triangle, and i directly precedes j when the boundary of this triangle is traversed in the anti-clockwise sense. We denote this quiver by $Q(\mathcal{T})$. Mutation of this quiver has a very elegant geometric interpretation in terms of flips.

Definition 5.2. Flips. Let $\mathcal{T}$ be a triangulation of a regular polygon P_n, and let i be an arc of $\mathcal{T}$. Then the *flip at* i is the triangulation $\mathcal{T}_i$ obtained by moving i to the other diagonal of the quadrilateral formed when i is deleted from $\mathcal{T}$.

It is fairly simple to prove $\mu_i(Q(\mathcal{T})) = Q(\mathcal{T}_i)$, so flips and mutations are on equal footing.

Recall that a quiver Q is of type A_n if it has the same underlying diagram as a Dynkin diagram of type A_n.

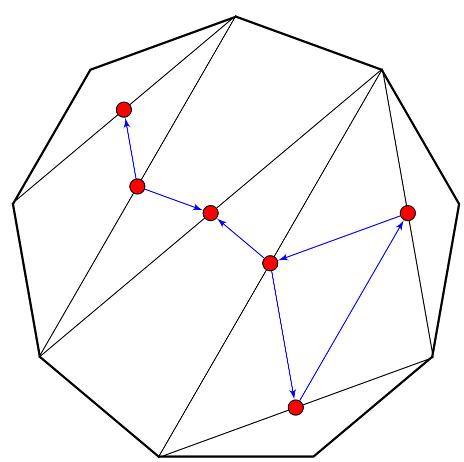

FIGURE 8. The assignment of a quiver to a triangulation.

Lemma 5.3. *Every quiver of type A_n may be obtained by a triangulation of a regular polygon.*

We may interpret this result by refining our previous point; we view triangulations and quivers of type A_n as being in bijective correspondence.

Lemma 5.4. *The number of triangulations of a convex polygon with n vertices is given by C_{n-2}, the $(n-2)$th Catalan number.*

Proof. This is a classical result; see for example [10]. □

We need to summarise some facts about associahedra before we prove Theorem 4.1. One should keep in mind Figure 6.

Lemma 5.5. *Let K_n be the $n-$associahedron. The vertices of K_n are in direct correspondence with distinct triangulations of P_{n+1}.*

Starting at a vertex on an associahedron then, a single flip of the corresponding triangulation to that vertex is equivalent to walking to an adjacent vertex on the associahedron.

Definition 5.6. Subpolygon. Let $n \geq k$ be integers. A *$k-$subpolygon* P_k (subpolygon for short) of P_n is a polygon obtained by choosing k vertices of P_n and inscribing a polygon on these vertices inside P_n.

We abuse notation from now on and use P_k to refer to convex polygons on k vertices, too. Note that a subpolygon partitions P_n into smaller sections; we refer to the region of P_n that does not lie in the inscribation as its *outside region.*

The key is the following:

Theorem 5.7. *Let K_n, K_k be associahedra with $n \geq k$. Distinct occurences of K_k in K_n are in correspondence with distinct occurences of P_{k-1} as a subpolygon of P_{n-1} with its outside region triangulated.*

Note this means that, with a fixed subpolygon P_{k-1}, distinct triangulations of the outside region correspond to distinct occurences of K_k in K_n.

Proof. Inscribe a subpolygon P_k of P_n and fix a triangulation of the outside region. Then, fixing a triangulation of P_k inside P_n, we obtain a full triangulation of P_n. This corresponds to a vertex of K_{n+1} by Lemma 5.5. By flipping arcs of the triangulation of P_k, we walk around an occurence of K_{k+1} inside K_{n+1}. Changing the triangulation of the outside region will lead to a new triangulation of P_n, so by flipping arcs in P_k again, we walk around a different occurence. Hence all the occurences of K_k in K_n are given by different ways of fixing P_{k-1} inside P_{n-1} and triangulating the outside region. □

We are now in a position to prove Theorem 4.1 from Section 4. We need to find the number of ways of fixing P_k inside P_n and triangulating the remain region. We will show the number of distinct occurences of K_k in K_n is given by

$$s_k^n = (n+1) \sum_{L \in P_{n-k}} (k)_{|L|-1} \Delta_L, \tag{5.1}$$

where P_{n-k} is the set of partitions of $n-k$, $(x)_n$ denotes the falling factorial of x up to n, and

$$\Delta_L = \prod_{l \in L} \frac{C_l}{\mathbf{R}(l)!} \tag{5.2}$$

where C_l is the lth Catalan number and $\mathbf{R}(l)$ is the number of parts of length l in L.

Proof. By virtue of Theorem 5.7 (and the subsequent remark), we need only consider the number of ways we may inscribe a k-gon into an n-gon, and triangulate the remaining region.

Let P_n be a regular polygon with n vertices. We now consider how we may inscribe a $k-$subpolygon P_k. We have $n-k$ vertices which are not vertices of P_k. Together these determine a partition of $n-k$, according to how many unused vertices are directly adjacent on the perimeter of P_n.

Fix a partition $L = (l_1, l_2, ..., l_m)$ of $n-k$, so we may write

$$n - k = \sum_{i=1}^{m} l_i,$$

where the l_i satisfy $l_1 \geq l_2 \geq ... \geq l_m \geq 1$. When we have chosen k vertices of P_n such that the spaces between the chosen vertices describe the partition L, the polygon P_n is partitioned into the subpolygon P_k and the smaller subpolygons $P_{l_i+2}, 1 \leq i \leq m$ comprising the outer region. Hence there are

$$\prod_{i=1}^{m} C_{l_i} \tag{5.3}$$

ways of triangulating the outer region by Lemma 5.4.

We now need to calculate the number of ways we can inscribe a $k-$gon into P_n such that the vertices of P_n in the outside region describe L. We need to decide how each vertex of P_n is allocated, so predetermine the locations of the outer vertices by appending to each group of outer vertices a vertex of P_k. This procedure uses m vertices of P_k, leaving $k-m$ vertices that are not attached to any groups of outer vertices, and hence are free to allocate (so between two groups of outer vertices there could be several vertices of P_k). The k groups of vertices can be assigned to P_n in $k!$ ways, and there are n ways we may choose a starting vertex for our assignment. We need to account for the fact that the $k-m$ ungrouped vertices are indistinguishable, so we divide by $(k-m)!$. And repeated parts of L will lead to indistinguishable groups, so we divide by $\prod_{l \in L} \mathbf{R}(l)!$, where $\mathbf{R}(l)$ is the number of repeats for a distinct part of L. Finally our procedure will count the same arrangement k times, so we need to divide by k. Putting this all together we get for our number of ways

$$n \frac{k!}{k!(k-m)!} \prod_{l \in L} \frac{1}{\mathbf{R}(l)!} = n \frac{(k-1)!}{(k-m)!} \prod_{l \in L} \frac{1}{\mathbf{R}(l)!}. \tag{5.4}$$

Now translate variables $n \to n+1$, $k \to k+1$. The number of ways a copy of K_k is inherited in K_n for a fixed partition L of $n-k$ is therefore given by

$$(n+1)\frac{(k+1)!}{(k+2-|L|)!}\prod_{l\in L}\frac{C_l}{\mathbf{R}(l)!} = (n+1)\frac{k!}{(k+1-|L|)!}\prod_{l\in L}\frac{C_l}{\mathbf{R}(l)!} \tag{5.5}$$

and by summing over all partitions of $n-k$ we obtain (4.1).

□

6. Potential extensions

We discussed finite type cluster algebras and seeds in this paper. The main investigation was seeds of type A_n. It would be interesting to see the analogue of the inheritance matrix formula as well as the Lagrange inversion formula analogue for seeds of different type. A sequel paper will discuss a similar combinatorial approach to seeds of other types, particularly D_n, via the more generalised framework of root systems.

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