Cluster Algebras and Associahedra

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Abstract

A **cluster algebra** (of geometric type) is a commutative ring $\mathcal{C}(Q)$, with distinguished generating set X, and whose algebraic and combinatoric structure depends only on a finite directed graph Q called a *quiver*. We obtain the generators as follows; to each vertex of the quiver, we attach a fixed element of X called a *cluster variable*, and apply systematic transformations called *exchanges* simultaneously to the quiver and attached cluster variables. In this project, we investigate the mechanisms that give rise to this algebra, and focus on the combinatorial structure of so-called *finite type cluster algebras*, in which the exchanging sequence described above always repeats.

Background

Quivers are finite directed graphs without loops or 2-cycles.

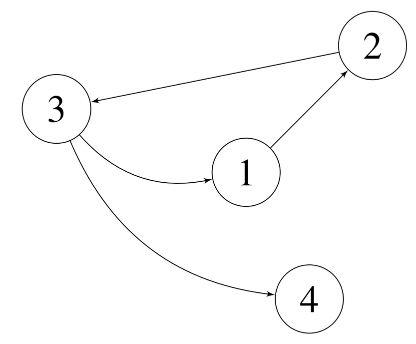

Fig. 1: A quiver.

On these graphs, we define an operation called *quiver mutation* at the kth vertex, denoted μ_k, defined on a quiver Q as follows:

1. Add shortcuts $i \to j$ for all sequences of arrows $i \to k \to j$.
2. Reverse any arrows at k.
3. Delete 2-cycles from Q.

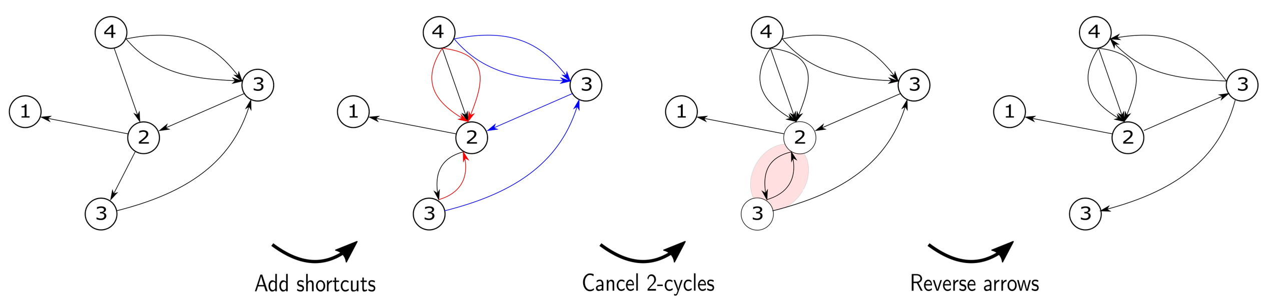

Fig. 2: An example of quiver mutation on the 3rd vertex.

We construct a graph of all the possible mutations of a quiver, called the *exchange graph*.

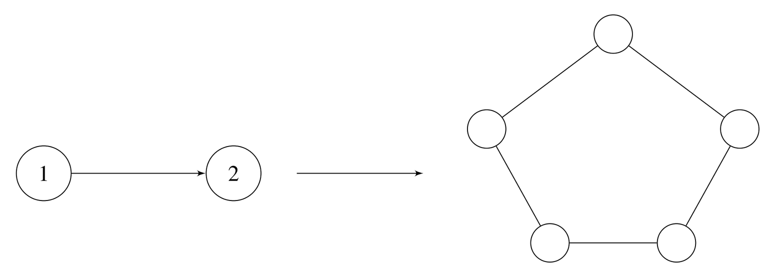

Fig. 3: The assignment of a quiver to its exchange graph.

We can construct a *cluster algebra* from the set up as follows; start with a fixed quiver Q and attach algebraically independent variables $x_1, x_2, ..., x_n$ to each of the vertices respectively.

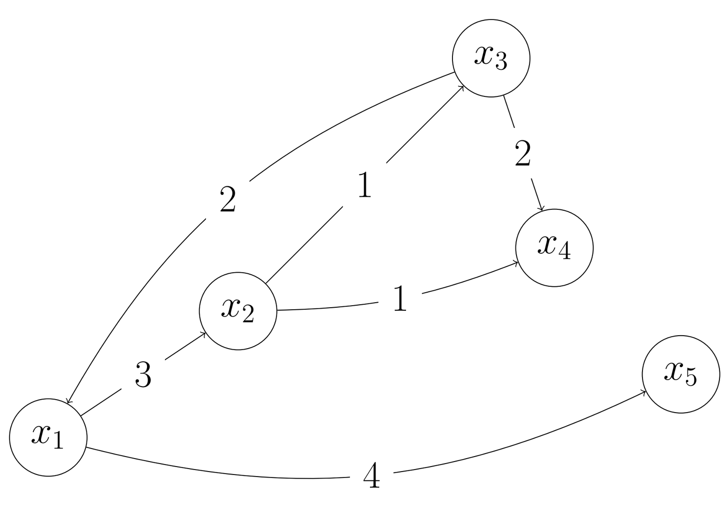

Fig. 4: Attaching cluster variables to a quiver.

Then, when mutating at vertex k, replace the cluster variable at k with x'_k, defined as follows:

$$x'_k = \frac{\prod_{i\to k} x_i + \prod_{k\to j} x_j}{x_k}. \tag{1}$$

We can then form the set X of generators of the cluster algebra, whose elements are all of the cluster variables we may obtain through mutation. Specifically, if we take $(\mathbf{x}, Q)$ to be the quiver with cluster variables attached, $\mathcal{C}(Q) = \langle X \rangle$, where

$$X = \bigcup_{(\mathbf{x}',Q')\sim(\mathbf{x},Q)} \{x'_1, ..., x'_n\}. \tag{2}$$

Quivers from triangulations

A class of quivers, called *quivers of type* A_n, are quivers mutation equivalent to an oriented chain of length n.

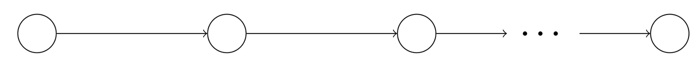

Fig. 5: A quiver of type A_n.

Mutation of quivers of this type have the elegant geometrical interpretation of *flips of triangulations*. Specifically, the type A_n quivers are those we may attach to a triangulation in the following way:

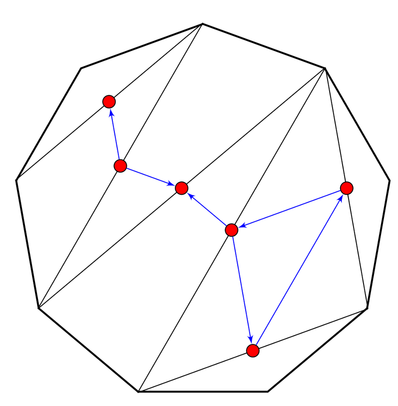

Fig. 6: The assignment of a quiver to a triangulation.

By choosing an arc of this triangulation and flipping to the other diagonal in the exterior quadrilateral, we obtain a new triangulation. The reconstruction of the quiver assigned to this triangulation is precisely the corresponding mutation. The *flip graph* of a triangulation in this case is exactly the exchange graph of one of the corresponding quivers, with cluster variables attached.

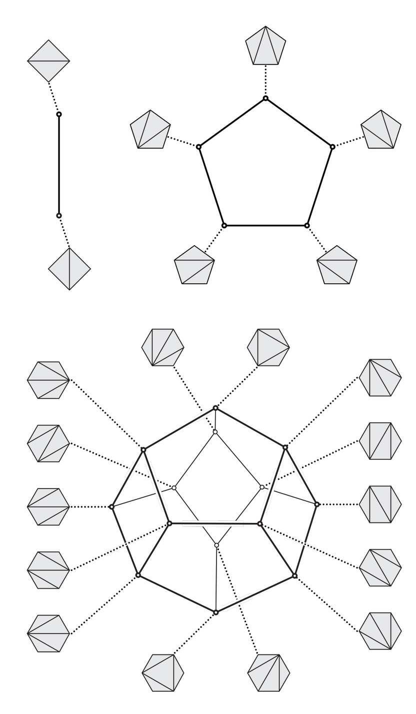

Fig. 7: Flip graphs of triangulations, and the corresponding polytopes.

Future Questions

Concluding this project left several avenues to explore:

1. Can we apply similar analyses to the other finite type quivers? For example, does the 'inverse power series' formula have an analogue for type D_n quivers?
2. How does mutation relate to isomorphisms of quivers? For a type X_6 quiver we may justify the exchange graph entirely in terms of the isomorphisms of X_6, but does a similar procedure apply to the other quivers?

Associahedra

The exchange graphs of the type A_n quivers are the nets of *associahedra*.

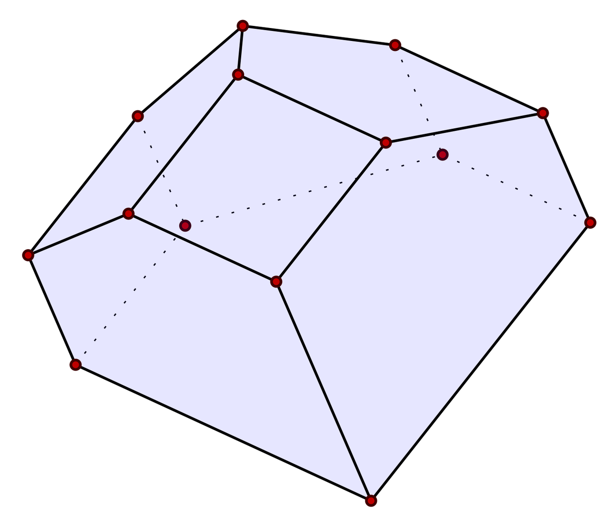

Fig. 8: The associahedron K_5.

One goal of the project was to understand the face structure of these shapes as much as possible. We have the *extended f-vector*:

$$f_{j-1} = \frac{1}{n-1}\binom{n-3}{j}\binom{n+j-1}{j+1} \tag{3}$$

Another result is the *inheritance matrix*, which records the distinct occurences of the associahedron K_k in the associahedron K_n as sub-polytopes:

$$s_k^n = (n+1)\sum_{L\in P_{n-k}} \frac{k!}{(k+1-|L|)!}\prod_{l\in L}\frac{C_l^{r(l)}}{r(l)!}, \tag{4}$$

where P_{n-k} is the set of partitions of $n-k$, C_l is the lth Catalan number, and $r(l)$ is the number of times part l is repeated in the partition of L. The precise face structure for a given associahedron is given by power series; let

$$f(x) = x + a_1x^2 + a_2x^3 + ... + a_nx^{n+1} + ... \tag{5}$$

and let $g(x) = x + b_1x^2 + b_2x^3 + ... + b_nx^{n+1} + ...$ be its inverse under composition. We have

$$b_n = \sum (-1)^{\sum n_i}\lambda(n_1, ..., n_k)a_1^{n_1}...a_n^{n_k} \tag{6}$$

where the coefficient $\lambda(n_1, ..., n_k)$ is the number of cells of the associahedron K_{n-1} that are isomorphic to the cartesian product $(K_0)^{n_1} \times ... \times (K_{k-1})^{n_k}$.

Finite type quivers

Shapiro et al. [1]: Quivers Q on at least 3 vertices are of finite mutation type iff they are one of the following:

1. Q associated with a triangulation of a two-dimensional bordered surface.
2. Q mutation-equivalent to one of the list of 11 quivers.

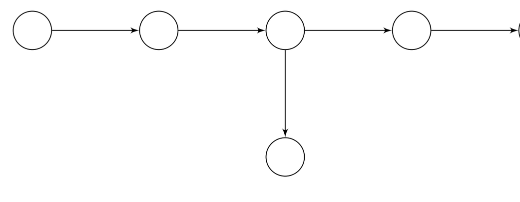

Fig. 9: Exceptional type quiver E_6.

I investigated these quivers, and gave the sizes of their mutation classes.

References

References

[1] Anna Felikson, Michael Shapiro and Pavel Tumarkin. Skew-symmetric cluster algebras of finite mutation type, 2008, J. Eur. Math. Soc. 14 (2012), 1135-1180; arXiv:0811.1703.